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THE EQUILIBRIUM BOUNDARY ELEMENT METHOD IN BOUNDARY-VALUE PROBLEMS OF ELASTICITY THEORY†

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An equilibrium boundary element method is proposed for solving boundary-value problems in the theory of elasticity, thermo-elasticity, the dynamical theory of elasticity, bar torsion calculations, and the bending of a plate. The idea is to use simultaneously the method of constructing bundles of functions which exactly satisfy the equilibrium equations, the boundary variational equations of mechanics, and the methods of discrete finite-element approximation. The variational method of constructing the resolving boundary equations ensures that the linear system is symmetric and easily coupled to the finite-element method. Since volume integrals are eliminated the dimensions of the problem are reduced by one, but, unlike the boundary element method, there is no need to know the fundamental solutions. The solution of some bar torsion and plate bending problems confirms the high numerical efficiency of the method.

The idea behind the equilibrium boundary element method (EBEM) [1] is the simultaneous use of discrete finite-element approximation principles [2], the methods of constructing basis function systems that exactly satisfy the equilibrium displacement differential equations [3–15], and the boundary variational equations of mechanics [8, 16–19]. Inside the body all the necessary equations are satisfied identically, hence the finite-element grid need only be constructed on the boundary. To determine the displacements of the boundary nodes, boundary variational equations are constructed which do not contain volume integrals. Thus, as in the boundary element method [4], one can reduce the dimensions of the problem by one, but unlike the latter, there is no need to use singular fundamental solutions. Here the variational method of construction ensures the symmetry of the resolving systems of linear equations and ease of coupling of EBEM to the finite-element method [1].

Unlike [16–19], non-singular basis polynomial solutions [3] are used below. This facilitates the numerical implementation of the method. Moreover, the boundary variational equations that are used can be obtained from a generalized Hu–Washizu variational principle [7], and the functional has the dimensions of energy. (Previous methods used were the collocation or least-squares methods [3, 16], "non-mechanical" functionals [17], and a truncated functional that only takes force boundary conditions into account [18, 19].) Finally, in many of these papers [3, 5, 16–20] few or no examples of the practical implementation of the method for solving boundary-value problems were given. In its concepts the present paper is close to [21], which used both polynomial fundamental solutions and similar "energetic" variational equations, but only applied to plane elasticity problems.

The high numerical efficiency of the EBEM is due to the low order of the resolving symmetric systems of linear equations, as is confirmed by a range of test problems.

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The existence of known representations of general solutions in terms of harmonic or other special functions for many classes of problems, the development of automated construction methods for equilibrium approximations, the possibility of optimal choice of the order and band-width of the resolving system of equations and the relative simplicity of the implementation make the EBEM a promising numerical method for solving static and dynamic boundary-value problems in the theory of elasticity for piecewise-homogeneous bodies, comparable in universality with the finite and boundary-element methods, and complementing them.

1. THE BOUNDARY-VALUE PROBLEM OF THE THEORY OF ELASTICITY

Suppose that a three-dimensional homogeneous anisotropic body occupies a volume V bounded by a surface S. Under the action of known body forces F_i and boundary loads p_i^*, specified on a part S_2 of the boundary, stresses σ_{ij} occur in the body which cause strains ε_{ij} and displacements u_i. Inside the body the stresses must satisfy the equilibrium differential equations

$$\sigma_{ij,j} + F_i = 0 \tag{1.1}$$

and on the boundary, the force boundary conditions

$$p_i = \sigma_{ij} n_j = p_i^* \quad \text{on} \quad S_2 \tag{1.2}$$

with the stresses and strains related by the generalized Hooke's law

$$\sigma_{ij} = d_{ijkl} \varepsilon_{kl} \tag{1.3}$$

and the deformations and displacements by the Cauchy formulae

$$\varepsilon_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k}) \tag{1.4}$$

Substituting (1.4) and (1.3) into (1.1) we obtain the displacement equilibrium equations

$$M_{ij} u_j + F_i = \tfrac{1}{2} d_{ijkl}(u_{k,lj} + u_{l,kj}) + F_i = 0 \tag{1.5}$$

where M_{ij} is a known second-order differential operator. Of the infinite number of solutions of these equations the correct ones are those that are identical on the boundary S_1 with specified displacements u_i^*

$$u_i = u_i^* \quad \text{on} \quad S_1 \tag{1.6}$$

Formulae (1.1)–(1.6) use the convention of summation over repeated indices, n_j are the direction cosines of the outward normals to the surface, and d_{ijkl} is the symmetric tensor of the elastic constants of the anisotropic material $(i, j, k, l = 1, 2, 3)$.

2. EQUILIBRIUM APPROXIMATION OF GENERAL FORM

To satisfy the equilibrium differential equations (1.5) exactly we represent the displacements by the following series with undetermined coefficients c_i

$$u_i = u_{0i} + c_1 u_{1i} + \ldots + c_n u_{ni} \tag{2.1}$$

where u_{0i} is some particular solution of the inhomogeneous equation (1.5)

$$M_{ij}u_{0j} + F_i = 0 \tag{2.2}$$

and the functions u_{ki} satisfy the appropriate homogeneous equations

$$M_{ij}u_{kj} = 0, \quad k = 1,\ldots,n \tag{2.3}$$

For any choice of coefficients c_i the approximation (2.1) is an equilibrium approximation, i.e. the displacements defined using it satisfy the equilibrium differential equations identically.

3. VARIATIONAL-BOUNDARY EQUATIONS

To solve the boundary-value problem approximately it remains to choose coefficients c_i so that the kinematic boundary conditions (1.6) and force boundary conditions (1.2) are optimally satisfied.

To do this one can use various methods. For example, to find the c_i it was suggested in [3] that the collocation method should be used, considering boundary conditions (1.2) and (1.6) at each of n chosen boundary points. However, firstly, the satisfaction of boundary conditions at several points does not guarantee that they will be satisfied between those points. Secondly, the collocation method leads to non-symmetric systems of linear equations. Thirdly, the conditioning of the resolving system of linear equations depends significantly on the choice of collocation points, and it is not in general clear how to choose them optimally.

With the method of least squares for determining the equilibrium expansion coefficients, it has been shown [5] that to obtain reliable solutions it is necessary to ensure not just "mean-square" convergence of the approximate to the exact functions, but also of their derivatives, which complicates the method considerably. Moreover, the conditioning of the resolving system of linear equations in the method of least squares is asymptotically much worse than in the collocation method [16].

Variational approaches derived from the well-known boundary variational equations of mechanics [1, 7, 8, 18, 19, 21] are the most hopeful. Firstly, they allow one to attach a mechanical interpretation to each equation. Secondly, for every problem one can explicitly construct an energy functional, which enables one to use the rich arsenal of the variational calculus. Thirdly, variational methods, as a rule, lead to symmetric systems of linear equations, which is highly desirable from a computational point of view.

To construct variational equations for the EBEM we write the Lagrange functional [6], using Lagrange multipliers [7] to add terms to it that include the error in the boundary displacements on the part of the surface S_1

$$\Pi(u_i) = \int_V \{\tfrac{1}{2}\varepsilon_{ij}\sigma_{ij} - u_iF_i\}dV - \int_{S_2} u_ip_i^*dS - \int_{S_1} p_i(u_i - u_i^*)dS \tag{3.1}$$

We integrate by parts the first term in the volume integral, having first expressed with the help of (1.4) the deformation in terms of the displacements

$$\Pi(u_i) = -\int_V \{\tfrac{1}{2}u_i\sigma_{ij,j} + u_iF_i\}dV + \int_S \tfrac{1}{2}u_ip_idS - \int_{S_2} u_ip_i^*dS - \int_{S_1} p_i(u_i - u_i^*)dS \tag{3.2}$$

We then use the equilibrium of the approximations for the displacements and constraints (1.3), (1.4). This leads to a well-known boundary condition functional [8]

$$\Pi(u_i) = -\int_V \tfrac{1}{2}u_iF_idV + \int_{S_2} u_i(\tfrac{1}{2}p_i - p_i^*)dS - \int_{S_1} p_i(\tfrac{1}{2}u_i - u_i^*)dS \tag{3.3}$$

Using the fact that for an exact solution of the boundary-value problem the functional (3.3) should have a stationary value, we arrive at the following variational equation [1, 7, 8]

$$\delta\Pi(u_i) = \int_{S_2} \delta u_i (p_i - p_i^*) dS - \int_{S_1} \delta p_i (u_i - u_i^*) dS = 0 \tag{3.4}$$

where δu_i and δp_i are the virtual displacements and loads at the boundary of the body.

The variational equation (3.4) does not contain volume integrals, which is its principal difference from the most commonly used variational equations of Lagrange, Reissner, Castigliano, etc. functionals in numerical methods of mechanics.

4. EQUILIBRIUM SUPERELEMENTS, THEIR BASIS FUNCTIONS, AND STIFFNESS MATRICES

We identify each homogeneous subdomain of the body with a single superelement, which in general has arbitrary shape and a variable number of boundary nodes. Suppose that an equilibrium approximation (2.1) has been constructed for some superelement. We choose $m = n/3$ nodes on the boundary of the superelement and substitute the coordinates of each of them into formula (2.1). We obtain as a result n linear equations for n coefficients c_i, from which one can express the c_i in terms of the displacements of the boundary nodes. We then substitute the c_i obtained into (2.1) and assume that the particular solution u_{0i} can be exactly reproduced by a linear combination of the functions u_{ki}. As a result we obtain the final representations for the displacements

$$u_i = N_{ir}\{u\}_r \tag{4.1}$$

where the $\{u\}_r$ are displacements of the superelement boundary nodes $(r = 1, \ldots, n)$, and the N_{ir} are basis functions for the superelement.

Note the difference between the basis functions of the superelement and the usual basis functions used in the finite-element method [2]. Firstly, each trio of basis functions N_{1r}, N_{2r}, N_{3r} satisfies the homogeneous differential equations for displacement equilibrium identically. Secondly, different components of the displacements are approximated by different basis functions. Thirdly, a certain component of the displacements is defined not only by the node values of that component, but by the node values of other components.

Having constructed the approximations for the displacements, from the subsequent formulae (1.2)–(1.4) one can construct approximations for the strains, stresses, loads and their variations

$$\begin{aligned}
\varepsilon_{kl} &= \tfrac{1}{2}(N_{kr,l} + N_{lr,k}) = B_{klr}\{u\}_r \\
\sigma_{ij} &= d_{ijkl}B_{klr}\{u\}_r = S_{ijr}\{u\}_r \\
p_i &= n_j S_{ijr}\{u\}_r = L_{ir}\{u\}_r \\
\delta u_i &= N_{ir}\{\delta u\}_r, \quad \delta p_i = L_{ir}\{\delta u\}_r
\end{aligned} \tag{4.2}$$

After substituting the required approximations into the variational equation (3.4) and using the arbitrariness of the node values of the virtual displacements we arrive at a system of linear equations for the equilibrium of the superelement

$$\begin{aligned}
K_{pr}\{u\}_r &= \{Q\}_p \\
K_{pr} &= \int_{S_2} N_{ip}L_{ir}dS - \int_{S_1} L_{ip}N_{ir}dS \\
\{Q\}_p &= \int_{S_2} N_{ip}p_i^* dS - \int_{S_1} L_{ip}u_i^* dS
\end{aligned} \tag{4.3}$$

with rigidity matrix K and load vector $\{Q\}$.

Combining similar systems of equations for all equilibrium superelements by the method described in [1] we obtain a general system of linear equations for the entire body. After solving it using (4.1) and (4.2) one can compute the displacements, strains and stresses at any point of the body.

Note that the stiffness matrix is symmetric. This is easy to prove, using the equilibrium of the displacement approximation and the well-known variational relation [7]

$$\int_S \delta u_i p_i dS = \int_S \delta p_i u_i dS \tag{4.4}$$

5. THE BOUNDARY ELEMENT MODEL OF THE BODY

To calculate the stiffness matrix and the load vector the surface of the superelement is partitioned into a set of simply-shaped boundary elements (BEs). The BEs are divided into two classes: BEs which belong to the part S_1 of the surface where the displacements (1.6) are specified, and BEs for the part S_2 of the surface where the loads (1.2) are specified. For BEs of each class the corresponding stiffness matrix and load vector components are computed numerically by integrating by quadratures. The matrix elements are then joined together as in the finite-element method.

Note that for body forces of complicated form it is not always possible to construct a particular solution u_{0i} for the inhomogeneous equation (2.2). In this case the equilibrium equation (2.1) has to be used without u_{0i}. Then in the resolving equations (4.3) the following additional vector of volume loads occurs

$$\{Q_F\}_p = \int_V N_{ip} F_i dV \tag{5.1}$$

To compute it, the interior of the superelement has to be partitioned into a set of finite elements, and approximations and integrations are performed on each of the latter using well-known methods [1, 2].

6. BAR TORSION PROBLEMS

In Saint-Venant boundary-value problems for the torsion of bars with complicated shape, from the solution of the differential equation

$$\partial^2\varphi / \partial x^2 + \partial^2\varphi / \partial y^2 = -2 \tag{6.1}$$

it is required to find the stress function $\varphi(x, y)$ that vanishes on the contour C of the cross-section of the bar

$$\varphi = 0 \quad \text{on} \quad C \tag{6.2}$$

An equilibrium approximation of type (2.1) can be written in the form

$$\varphi(x, y) = -(x^2 + y^2) / 2 + c_1 H_1 + \ldots + c_n H_n \tag{6.3}$$

where the H_i are harmonic polynomials satisfying the two-dimensional Laplace equation, for example power polynomials of the form [3]

$$H_1 = 1,\quad H_2 = x,\quad H_3 = y,\quad H_4 = x^2 - y^2,\quad H_5 = 2xy$$

$$H_6 = x^3 - 3xy^2,\quad H_7 = 3x^2y - y^3,\quad H_8 = x^4 - 6x^2y^2 + y^4$$

$$H_9 = 4x^3y - 4xy^3, \ldots \tag{6.4}$$

The resolving system of linear equations for finding the approximation coefficients has the form

$$\int_C \frac{\partial H_i}{\partial n} H_j dC \cdot c_j = \int_C \frac{\partial H_i}{\partial n} \frac{x^2 + y^2}{2} dC \tag{6.5}$$

where, to determine the coefficient c_1, it is convenient to use the method of least squares [1].

The BEs are curvilinear segments.

7. BENDING PROBLEMS FOR THIN ISOTROPIC PLATES

The bending of a plate of stiffness D under the action of a load p is governed by the well-known Sophie Germain equation [9]

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{p}{D} = 0 \tag{7.1}$$

The equilibrium approximation is given by the sum of the particular solution w_0, which is usually easy to construct, and the biharmonic function w_1, which with the help of Almansi's formula [3] can be expressed in terms of two harmonic functions φ_1 and φ_2

$$w_1 = \varphi_1 + (x^2 + y^2)\varphi_2 \tag{7.2}$$

each of which can be represented as a series in the harmonic polynomials (6.4).

The BEs are curvilinear segments.

8. THREE-DIMENSIONAL PROBLEMS WITH AN ELASTIC ISOTROPIC BODY

According to the Papkovich–Neuber representation [10, 11] the general solution of the system of homogeneous differential equations (1.5) can be represented in terms of three harmonic functions φ_1, φ_2, φ_3 as follows (ν is Poisson's ratio)

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} - \alpha \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \\ \partial / \partial z \end{Bmatrix} (x\varphi_1 + y\varphi_2 + z\varphi_3),\quad \alpha = \frac{1}{4(1-\nu)} \tag{8.1}$$

Each of the functions φ_i can be represented as a series in known particular solutions of the three-dimensional Laplace equation, for example, in terms of the complete system of three-dimensional harmonic polynomials [3]

$$1; x, y, z;\quad z^2 - x^2, x^2 - y^2, xy, yz, zx;\quad z^3 - 3x^2z,$$

$$-x^2z + y^2z, -3xz^2 + x^3, -x^3 + 3xy^2, yz^2 - x^2y, xyz; \ldots \tag{8.2}$$

As well as the above polynomial representations, one can also use other systems of harmonic functions. For example, the completeness and linear independence of the functions

$$\{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2\}^{-1/2} \tag{8.3}$$

was proved in [16], where x_i, y_i, z_i are coordinates of some auxiliary points, usually chosen on a closed contour enclosing the interior of the body.

For a three-dimensional body the BEs can be triangular or rectangular curvilinear isoparametric elements [1, 4].

9. PLANE PROBLEMS IN THE THEORY OF ELASTICITY

Representations of the displacements u_x and u_y in terms of two harmonic functions can be obtained from (8.1) if one puts $u_z=0$ $\varphi_3=0$, and for a plane loaded stated $\alpha^{-1}=4(1-2\nu)/(1-\nu)$. Similar representations for axisymmetric problems are given in [12].

10. PROBLEMS IN ELASTIC THEORY OF AN ANISOTROPIC BODY

Representations similar to those given above in terms of harmonic functions for general solutions of boundary-value problems are only known for certain special anisotropic cases [13]. In the general anisotropic case, to construct equilibrium approximations in the class of power polynomials one has to use the method of undetermined coefficients [1, 3], the essence of which is as follows. We represent the displacements in the form of power series

$$u_i = a_{ijkl}x^j y^k z^l \tag{10.1}$$

and substitute them into the homogeneous equations (1.5). After differentiation, each of the three equations becomes the condition for some polynomial with coefficients depending linearly on the a_{ijkl} to be zero

$$b_{mnr}x^m y^n z^r = 0 \tag{10.2}$$

The set of equations $b_{mnr}=0$ determines the constraint relations between the coefficients a_{ijkl}, from which it is necessary to choose n independent ones and denote them by $c_1, \ldots, c_n$ Taking account of these conditions in (10.1) we can obtain $3n$ functions u_{ki} of the equilibrium approximation (2.1). The process of constructing equilibrium approximations by the method of undetermined coefficients can easily be carried out on a computer.

11. THERMO-ELASTICITY PROBLEMS

The particular solution u_{0i} of the thermo-elasticity problem can be obtained using the Papkovich–Goodier representation [10]

$$u_{0i} = \partial\Phi / \partial x_i \tag{11.1}$$

where Φ is a function satisfying Poisson's equation

$$\frac{\partial^2\Phi}{\partial x_i \partial x_i} = \frac{1+\nu}{1-\nu}\alpha T \tag{11.2}$$

α is the coefficient of thermal expansion, and T is the known temperature of the body.

12. DYNAMICAL PROBLEMS IN THE THEORY OF ELASTICITY

Using the well-known Green–Lamé representation [14]

$$\mathbf{u} = \operatorname{grad} \varphi + \operatorname{rot} \psi \tag{12.1}$$

the displacements of an isotropic body can be expressed in terms of a scalar and vector potential

$$\frac{\partial^2 \varphi}{\partial t^2} - c_1^2 \frac{\partial^2 \varphi}{\partial x_i \partial x_i} = 0, \quad \frac{\partial^2 \psi}{\partial t^2} - c_2^2 \frac{\partial^2 \psi}{\partial x_i \partial x_i} = 0, \quad \operatorname{div} \psi = 0 \tag{12.2}$$

where c_1 and c_2 are known velocities of propagation of longitudinal and transverse waves in an infinite medium and t is the time. In the general case, the functions φ and ψ can be represented by series of known particular solutions of the wave equation [3, 14, 15], for example

$$\varphi = A\cos(\omega t \pm k_i x_i + \alpha_i), \quad k_i k_i = k^2 = \omega^2 / c_1^2 \tag{12.3}$$

13. EXAMPLES OF CALCULATIONS OF BAR TORSION

The advantages of the EBEM compared with the other numerical methods are clearly illustrated in bar torsion problems. In particular, cross-section shapes exist (circle and equilateral triangle) for which the exact solution of the boundary-value problem (6.1), (6.2) can be obtained exactly for one equilibrium superelement with a small number of boundary nodes. For example, for an equilateral triangle of side α the exact solution [6]

$$\varphi(x,y) = \frac{2a^2}{27} - \frac{x^2+y^2}{2} + \frac{x^3 - 3xy^2}{2a}$$

can be reproduced on a triangular superelement with seven boundary nodes.

Unlike the preceding example, the exact solution

$$\varphi(x,y) = -\tfrac{4}{5}(\tfrac{1}{4}x^2 + y^2 - 1)$$

for an ellipse [6] with semi-axes 2 and 1 cannot be exactly generated by the series (6.3). However, as shown by calculations, for an increasing number of boundary nodes m the approximate solution converges monotonically to the exact solution. For example, at the centre of the ellipse, instead of the exact value $\varphi = 0.8$ for $m = 4, 8, 16$ we obtain the values 0.683, 0.739, 0.785, respectively.

Figure 1 shows graphs illustrating the stable convergence of the numerical solutions to the exact solutions when the number of BEs increases, for the torsion problem for bars with square $1 \leqslant x,\ y \leqslant 1$ and flag-shaped (Fig. 2) cross-sections.

For example, consider the solution for a square with $m = 12$ BEs

$$\varphi(x,y) = -\tfrac{1}{2}(x^2+y^2) + 0.5885 - 0.0972(x^4 - 6x^2y^2 + y^4) \tag{13.1}$$

This is an approximate solution for a square. However, there is a shape, approximately square, for which this solution is exact. The equation for this shape is obtained from (13.1) by replacing $\varphi(x, y)$ by zero. By increasing the number of BEs, the shapes for which the solutions obtained are exact tend closer and closer to a square. This is shown in Fig. 3.

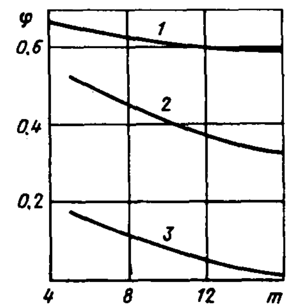

FIG. 1.

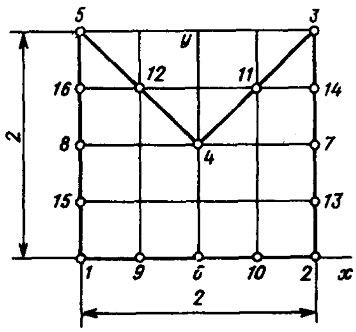

FIG. 2.

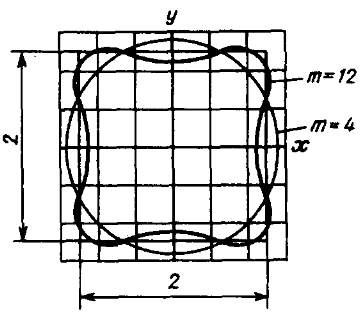

FIG. 3.

14. EXAMPLES OF CALCULATIONS FOR THE BENDING OF CLAMPED PLATES

Table 1 gives data illustrating the convergence, as the number m of BEs increases, of the deflection Δ and bending moments M_r and M_x of circular $(r=1)$, square $(a=b=2)$ and rectangular $(a=1,\ b=2)$ plates to the solutions obtained in [9] by analytic methods of expansion in trigonometric functions $p/D=1,\ \nu=0.3$.

The efficiency of the EBEM is also confirmed by calculations for more complicated plates. For example, effective convergence with an error of less than 1% is observed with $m=36$ for a plate in the shape of an isosceles trapezium with base lengths 3 and 1 and an angle of 45° between the lower base and the lateral side. Here the maximum deflection and bending moment are observed at the midpoint of the centre line of the trapezium and are equal to 0.1517 and 1.1641, respectively, when $p/D=1,\ \nu=0.3$, which differs by less than 3% from the result obtained in [20] by the R-function method.

We will now consider the bending by a uniform load of a rhomboidal plate fixed at its edges for various ratios a/b of the lengths of the diagonals. The boundary of the plate is partitioned uniformly into 32 linear BEs. Table 2 shows the values of the deflection Δ and bending moments M_x and M_y at various points on the major diagonal of the rhombus (x axis) when $b=1,\ p/D=64,\ \nu=0.3$. Analysis of the results shows that as the ratio of the diagonals increases, there is an increase in the maximum deflection and bending moment, both of which occur at the centre of the plate. Note that at the corner point $(a, 0)$ the bending moment is undefined, because the derivative is undefined. To calculate bending moments correctly in the neighbourhood of the corner points one must take into account the singular behaviour of solutions at irregular boundary points, which is beyond the scope of this paper.

The EBEM is constructed using the variational equations of mechanics and hence guarantees the exact satisfaction of differential equations inside the plate and the integrated (non-exact!) satisfaction of the boundary conditions. The deflections calculated at the fixed boundary are not identical with the exact zero values. However, the error of the deflections at the boundary is fairly small and hence can be ignored. In particular, when $a/b=1$; 2; 4 the maximum deflection at the boundary does not exceed 3, 5, 8%,

TABLE 1

m	Circle		Square		Rectangle	
	Δ	M_r	Δ	M_x	Δ	M_x
4	0.622	0	0.689	0	3.59	0
8	0.812	0.902	0.689	1.17	2.27	1.90
16	0.949	0.975	0.825	1.07	0.94	1.64
20	0.968	0.984	1.03	1.01	1.13	1.11
24	0.977	0.989	1.03	1.01	1.06	1.05
30	0.985	0.993	1.01	1.00	1.00	1.00

TABLE 2

Point coordinates	$a/b=1$			$a/b=2$			$a/b=4$		
	Δ	M_x	M_y	Δ	M_x	M_y	Δ	M_x	M_y
(0.0)	0.327	2.956	2.956	0.959	4.223	5.686	1.604	3.845	8.101
(1a/4.0)	0.261	2.001	2.672	0.661	2.128	4.837	0.958	1.993	6.098
(2a/4.0)	0.119	−0.134	1.807	0.183	−0.655	2.564	0.115	−0.217	2.323
(3a/4.0)	0.018	−1.374	0.433	0.012	−0.239	0.612	0.028	0.493	0.978
(4a/4.0)	0.002	1.212	−0.941	0.003	−0.422	0.647	0.008	−2.001	−3.260

respectively, of the maximum deflection at the centre of the plate. The correct modelling of the plate stiffness is confirmed by comparing the maximum deflections with finite-element calculations when the plate is decomposed into 16 hybrid finite elements [1]. For $a/b=1$; 2; 4 the latter give the values 0.320; 0.859; 1.30, respectively.

Thus, for plate calculations with various not-too-complicated shapes it is sufficient to restrict oneself to 30–40 BEs, and by taking possible symmetry into account the number of BEs can be reduced considerably. Given that the time for a single calculation on an IBM PC/AT is no more than 5 minutes, one can say that a simple and efficient method has been created for calculating the bending of a plate of complicated shape.

15. VERSIONS OF THE APPLICATION OF THE METHOD

To calculate the deformation of homogeneous bodies whose shape is not too complicated one can use a single equilibrium superelement. For piecewise-homogeneous or homogeneous bodies of complicated shape one must use several elements and ensure they are matched [1]. Here, by varying the number and dimensions of the superelements one can achieve an optimal combination of the order and width of the strip of the global system of linear equations [1]. The limit in which the superelement decreases in size to the dimensions of a finite element reduces to the method of equilibrium finite elements [1]. A combination of the EBEM method with the finite-element method, in which the homogeneous part of the body is represented by an equilibrium superelement, and the remainder by finite elements, is very promising.

In conclusion we remark that the EBEM, like the BEM, cannot be used to solve boundary-value problems described by systems of differential equations with variable coefficients, for example, to analyse plates or shells of variable thickness. The use of the method to calculate natural oscillations and non-linear boundary-value problems is very problematical, because one cannot just use boundary-element grids. One must also bear in mind that one of the disadvantages of the method, inherent in all variational methods, is that solutions near boundary corner points or points where the type of boundary condition changes are poorly modelled by smooth harmonic polynomials. For this one must use more complicated methods that take account of singularities.

The EBEM was implemented as part of the ASTRA Computer-aided System of Three-dimensional Equilibrium Analysis [1].

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